Visco-Elastic Theory of the Deformation of a Confined Aquifer

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Bull. Disas. Prev. Res. Inst., Kyoto Univ., Vol. 19, Part 1, No. 150, Aug. 1969

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(Manuscript received May 31, 1969)

Abstract

The author derived a dynamic theory for the deformation of a granular solid saturated with a liquid, assuming that the liquid filling up the pore space is a Newtonian viscous fluid and that the skeleton constituted by solid particles is a linear visco-elastic solid. The theory consists of the three fundamental equations, that is, the equations of motion of the liquid and the skeleton and the equation of continuity between the particles and the liquid. In a case where the particle and liquid are taken to be incompressible and the deformation of soil takes place on a quasi-static process, these equations are accepted as the theory of three-dimensional consolidation, including Terzaghi's well-known equation as a special case, and are also recognized as the basic equations of motion of confined ground water in a visco-elastic aquifer. A theoretical example will be shown for the rheological deformation of an infinite confined aquifer with uniform thickness caused by pumping up water at a constant rate.

1. Introduction

It is said that the subsidence of a ground surface is based on the contraction of the soil layer caused by the depression of the pore pressure. The conception of pore pressure was first introduced by K. v. Terzaghi and soil mechanics has made great advances through its conception. He considered that the soil particles, being more or less bound to each other by the sticking force, constitute a skeleton of soil with elastic properties and that the skeleton supports the external burden together with cooperation of the water filling up the pore space between the particles, and he successfully solved the settlement of the soil layer with the good idea that a contraction of the soil depends on the rate of squeezing out of pore which neccessarily brings about the decrease of the pore water pressure. But he treated only a one-dimensional problem under constant load with a quasi-static method.

In Oct. 1940, M. A. Biot[1] published the theory of three-dimensional consolidation and developed the treatment of soil deformation for any arbitrary load variable with time. In Sep. 1963, M. Mikasa[2] published a useful theory of soft layer consolidation showing many suitable examples, especially taking account of finite strain. But they treated only an elastic problem in the same way as Terzaghi.

Although much attention has been paid to damage due to subsidence and persevering efforts have been made to elucidate its mechanism on the basis of the above theories, the results of many investigations indicate the nonelastic deformation of the soil and the unfitness of the elastic theory for the quantitative explanation of the settlement. Of course, the deformation of the soil layer could not be a simple rheological model owing to the irregularity of the alluvial structure and the complexity of the mechanical character of soil.

We shall now derive in this paper the visco-elastic deformation of a confined aquifer as the first step in the rheological research of subsidence.

2. Derivation of Fundamental Equations

Soil particles constitute the skeleton of the soil matrix, pushing and rubbing each other's contact portions against an external burden. Owing to the complexity of its structure, however, one could not expect the direct treatment of forces acting on each particle. In the same situation, it would also be quite impossible to deal quantitativey with the motion of pore water attending to the tortuous and irrregular pore space. Therefore we are obliged to consider the representation of motion averaged over a volume element of soil, which is taken to be large enough compared to the size of the pores, so that it may be treated as homogeneous, and at same time small enough compared to the scale of macroscopic phenomena in which we are interested, so that it may be considered as infinitesimal in the mathematical treatment. It will be sufficient in soil mechanics to consider the average conditions over the volume of soil in the above sense.

(a) Equation of motion of pore water

The motion of water in pores is governed by the hydrodynamic equation of viscous fluid. We regard the pore water as a Newtonian fluid and denote by $\boldsymbol{V}$ (V_1, V_2, V_3) the particle velocity of pore water. The equation of motion of pore water is expressed by

$$\frac{D\boldsymbol{V}}{Dt}=\boldsymbol{X}-\frac{1}{\rho}\operatorname{grad} p-\left(\frac{1}{3}\eta-\kappa\right)\operatorname{grad}\theta+\eta\nabla^2\boldsymbol{V} \tag{1.1}$$

where t is time, $\boldsymbol{X}$ is external body force, ρ and p are the density and pressure of water, respectively, θ is the divergence of water flow, and η and κ are the kinematic viscosity of shear and bulk respectively; the dependences of which on density ρ are assumed to be slight.

Consider a unit volume of the soil matrix in the sense stated above. Integrating Eq. (1.1) over the pore space σ of the unit volume and using the following notations:

$$\boldsymbol{U}\equiv\iiint_\sigma \boldsymbol{V}dv, \qquad P\equiv\frac{1}{\sigma}\iiint_\sigma p\,dv \tag{1.2), (1.3}$$

we have

$$\frac{D\boldsymbol{U}}{Dt}=\sigma\boldsymbol{X}-\frac{\sigma}{\rho}\operatorname{grad} P-\left(\frac{1}{3}\eta-\kappa\right)\iiint_\sigma \operatorname{grad}\theta\,dv+\eta\iiint_\sigma\nabla^2\boldsymbol{V}\,dv \tag{1.4}$$

where $\boldsymbol{U}$ is called "Darcy's velocity" or the "specific flow rate" and P is the "pore pressure" proposed by Terzaghi.

The pore pressure P is generally taken to be thermodynamic pressure and is determined by the density and temperature of the water. In constant temperature, we can write

$$\log\frac{\rho}{\rho_0}=\beta(P-P_0) \tag{1.5}$$

where β is called isothermal compressibility. In the foregoing, ρ_0 and P_0 are the density and pressure in some reference state, say a state at rest. In the case where external body force $\boldsymbol{X}$ is gravitational force, it is convenient to introduce the quantity φ which is termed the "piezometric head"

$$\varphi \equiv \frac{P}{\rho g} + x_3 \tag{1.6}$$

where x_3- axis is taken as positive upward.

Appropriate expression to the last term in Eq. (1.4) is given by referring to Darcy's law governing the flow of water in a porous medium. He postulated the viscous force acting on water to be proportional to the flow velocity and introduced, as the proportional constant, the physical quantity k which is called the coefficient of permeability of the soil, and according to his expression, the viscous force is written by

$$\boldsymbol{F} \equiv \eta \iiint_\sigma \nabla^2 \boldsymbol{V} dv \equiv - \frac{\sigma g}{k} \iiint_\sigma \boldsymbol{V} dv$$

In the next paragraph we shall treat the motion of soil particles together with the flow of pore water, and so it may be reasonable to assume that the viscous force $\boldsymbol{F}$ was proportional to the relative motion of water to soil particles, viz. $\left(\boldsymbol{V} - \frac{\partial \boldsymbol{u}}{\partial t}\right)$ where $\boldsymbol{u}$ is the mean displacement of soil particles as seen latter. From our assumption, it is possible to express its force as follows

$$\boldsymbol{F} \equiv - \frac{\sigma g}{k} \iiint_\sigma \left(\boldsymbol{V} - \frac{\partial \boldsymbol{u}}{\partial t}\right) dv = - \frac{\sigma g}{k} \left(\boldsymbol{U} - \sigma \frac{\partial \boldsymbol{u}}{\partial t}\right) \tag{1.7}$$

Pore water is regarded to be almost incompressible in engineering practice. In this case, inserting the expressions (1.5), (1.6) and (1.7) into Eq. (1.4) and neglecting the inertia part of acceleration,

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sigma g \left\{ \text{grad } \varphi + \frac{1}{k} \left(\boldsymbol{U} - \sigma \frac{\partial \boldsymbol{u}}{\partial t} \right) \right\} = 0 \tag{1.8}$$

Eq. (1.8) is, of course, reduced to Darcy's law if the soil particle has no motion and the flow of water is not accelerated.

(*b*) *Equation of motion of soil skeleton*

we shall now pay attention to the motion of the soil skeleton. Terzaghi, Biot and Mikasa assumed the elastic isotropy of stress-strain relations for the soil skeleton. While we also accept the isotropy in order to avoid the trouble of mathematical presentation, we had better give up the elastic property of soil with reference to the results of many investigations postulating the nonelastic deformation of soil.[3] On the other hand, it may be clear that the non-linear relation between stress and strain make it difficult to analyse the deformation quantitatively. We now regard it tentatively as a linear visco-elastic relation.[4]

Consider again the volume element in the sense stated previously and take the average over the actual displacement $\boldsymbol{v}$ of soil particles contained in that volume. We define it as the displacement of skeleton, $\boldsymbol{u}$, that is

$$\boldsymbol{u} \equiv \frac{1}{1-\sigma} \iiint_{(1-\sigma)} \boldsymbol{v} dv \tag{1.9}$$

Supposing that the difference $(\boldsymbol{v}-\boldsymbol{u})$ produces only a minor effect on the stress on the skeleton viz. effective stress and assuming the strain to be infinitesimally small, the strain on the skeleton is given by tensor e_{ij}

$$e_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{1.10}$$

Corresponding to this, the effective stress σ_{ij} is exerted on the skeleton. According to the assumption of linear visco-elasticity,[4] stress σ_{ij} is represented, as positive in compression, by

$$\left(1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}\right)\sigma_{ij}=-\lambda\left(1+\sum_{p=1}^{l}\alpha_p\frac{\partial^p}{\partial t^p}\right)\delta_{ij}\Theta-2\mu\left(1+\sum_{p=1}^{m}\beta_p\frac{\partial^p}{\partial t^p}\right)e_{ij} \tag{1.11}$$

where δ_{ij} is the Kronecker notation and Θ is the dilatation of the soil skeleton, that is

$$\Theta\equiv\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3} \tag{1.12}$$

Introducing the operation with respect to time

$$\mathfrak{L}\equiv\lambda\frac{1+\sum_{p=1}^{l}\alpha_p\frac{\partial^p}{\partial t^p}}{1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}},\qquad \mathfrak{M}\equiv\mu\frac{1+\sum_{p=1}^{m}\beta_p\frac{\partial^p}{\partial t^p}}{1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}} \qquad (1.13),\ (1.14)$$

we can write the stress force per unit cubic element of soil as follows

$$\frac{\partial\sigma_{ij}}{\partial x_j}=-\ (\mathfrak{L}+\mathfrak{M})\frac{\partial\Theta}{\partial x_i}-\mathfrak{M}\nabla^2u_i \tag{1.15}$$

Furthermore, the skeleton is pushed by the pore pressure of the surrounding water. This pressure action $\boldsymbol{f}_1$ may not produce any shearing strain by reason of the assumed isotropy and will be expressed by

$$\boldsymbol{f}_1=\iint_{(1-\sigma)surf}p\boldsymbol{n}da=-\iiint_{(1-\sigma)}\text{grad }p dv=-(1-\sigma)\text{grad }P \tag{1.16}$$

per unit volume of soil, where $\boldsymbol{n}$ is the outward unit vector normal to the surface element da of solid space.

In addition to the above forces, the skeleton tends to be dragged by the flow of pore water in its direction through the reaction of the viscous force acting on the pore water. This drag force $\boldsymbol{f}_2$ will be expressed by

$$\boldsymbol{f}_2=-\rho\frac{\sigma g}{k}\left(\sigma\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{U}\right) \tag{1.17}$$

We can thus establish the equation of motion of the soil skeleton, that is

$$\begin{aligned}(1-\sigma)\rho_s\frac{\partial^2\boldsymbol{u}}{\partial t^2}&=(1-\sigma)\boldsymbol{X}-\nabla\cdot\sigma+\boldsymbol{f}_1+\boldsymbol{f}_2\\&=(1-\sigma)\boldsymbol{X}+(\mathfrak{L}+\mathfrak{M})\text{grad}\Theta+\mathfrak{M}\nabla^2\boldsymbol{u}-(1-\sigma)\text{grad}P+\rho\frac{\sigma g}{k}\left(\boldsymbol{U}-\sigma\frac{\partial\boldsymbol{u}}{\partial t}\right)\end{aligned} \tag{1.18}$$

where ρ_s is the density of soil particles and $\boldsymbol{X}$ is an external body force. In almost all cases with which we are concerned, $\boldsymbol{X}$ is a gravitational force. Expressing the vertically upward unit vector by $\boldsymbol{k}$, we have

$$\begin{aligned}(1-\sigma)\rho_s\frac{\partial^2\boldsymbol{u}}{\partial t^2}=&-(1-\sigma)\rho_s g\boldsymbol{k}+(\mathfrak{L}+\mathfrak{M})\text{grad}\Theta+\mathfrak{M}\nabla^2\boldsymbol{u}\\&-(1-\sigma)\text{grad}P+\rho\frac{\sigma g}{k}\left(\boldsymbol{U}-\sigma\frac{\partial\boldsymbol{u}}{\partial t}\right)\end{aligned} \tag{1.19}$$

(*c*) *Equation of mass continuity*

Finally, we shall derive the equation of mass continuity per unit volume of soil. Suppose that a skeleton in any volume of soil had porosity σ_0 at an instance of no dilatation $\Theta=0$, and that the particles in it had density ρ_{s0} at that time. Because the skeleton under consideration is to be framed by the same particles at any instance, the mass of the skeleton must be conserved, that is

$$\rho_s(1-\sigma)(1+\Theta)=\rho_{s0}(1-\sigma_0) \tag{1.20}$$

since the dilatation Θ represents the volume increase of soil skeleton per unit initial volume and ρ_s is the density of particles at dilatation Θ. The volume of the skeleton is varied with time by external force and consequently, porosity σ is also varied. Eq. (1.20) gives us the relation between their time rates. As our subject is soil fully saturated with pore water, the change of pore volume results in the flow of pore water into or out of the volume element. This situation is represented by

$$\frac{\partial}{\partial t}(\sigma\rho) = -\mathrm{div}(\sigma\rho \boldsymbol{V}) = -\mathrm{div}(\rho \boldsymbol{U}) \tag{1.21}$$

where ρ is the density of water, $\boldsymbol{V}$ is the particle velocity of water and $\boldsymbol{U}$ is the specific flow rate as defined previously.

Combining Eq. (1.20) with Eq. (1.21), we make

$$\frac{\partial}{\partial t}\left\{(1-\sigma)\rho_s+\sigma\rho\right\} = \frac{\partial}{\partial t}\left\{\frac{(1-\sigma_0)\rho_{s0}}{1+\Theta}\right\} - \mathrm{div}(\rho \boldsymbol{U}) \tag{1.22}$$

This is an equation which we expected to derive. Relation of ρ_s to ρ_{s0} may be obtained from the consideration of the compression of soil particles due to the effective stresses σ_{ij} and pore pressure P, although we have little knowledge about it at present. However, in general, the compressibilities of soil particles and water are small. Assuming both densities to be constant we rewrite Eq. (1.22) as

$$(1-\sigma_0)\frac{\partial\Theta}{\partial t} + \mathrm{div}\, \boldsymbol{U} = 0 \tag{1.23}$$

with good approximation neglecting the small quantity of order Θ^2.

Approximating the values σ in Eqs. (1.8) and (1.19) to the value σ_0 after Eq. (1.23) and summarizing the fundamental eqations in the case where the external body force is only gravitational force and the soil particles and pore water are incompressible, we have

$$\begin{aligned}(1-\sigma_0)\,\rho_{s0}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = &-(1-\sigma_0)(\rho_{s0}-\rho_0)g\boldsymbol{k} + (\mathfrak{L}+\mathfrak{M})\mathrm{grad}\Theta + \mathfrak{M}\nabla^2\boldsymbol{u} \\ &-(1-\sigma_0)\,\rho_0 g\ \mathrm{grad}\varphi + \rho_0\frac{\sigma_0 g}{k}\left(\boldsymbol{U}-\sigma_0\frac{\partial \boldsymbol{u}}{\partial t}\right)\end{aligned} \tag{a}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sigma_0 g\left[\mathrm{grad}\ \varphi + \frac{1}{k}\left(\boldsymbol{U}-\sigma_0\frac{\partial \boldsymbol{u}}{\partial t}\right)\right] = 0 \tag{b}$$

$$(1-\sigma_0)\frac{\partial\Theta}{\partial t} + \mathrm{div}\, \boldsymbol{U} = 0 \tag{c}$$

where

$$\Theta \equiv \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \qquad \varphi \equiv \frac{P}{\rho_0 g} + x_3 \tag{d), (e}$$

$$\mathfrak{L} \equiv \lambda\frac{1+\sum_{p=1}^{l}\alpha_p\frac{\partial^p}{\partial t^p}}{1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}}, \qquad \mathfrak{M} \equiv \mu\frac{1+\sum_{p=1}^{m}\beta_p\frac{\partial^p}{\partial t^p}}{1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}} \tag{f), (g}$$

In the remainder, we shall omit the subscript "$_0$" from the respective notations.

3. Derivation of Fundamental Equations of Subsidence

In this section, we shall seek the fundamental equations for the deformation of a confined aquifer caused by ground water flow.

Let us first examine a static equilibrium state of the skeleton with a steady flow of pore water. From Eqs. (a) (b) and (c), we have

$$0=-(1-\sigma)(\rho_s-\rho)g\boldsymbol{k}+(\mathfrak{L}+\mathfrak{M})\mathrm{grad}\Theta+\mathfrak{M}\nabla^2\boldsymbol{u}-\rho g \text{ grad } \varphi \tag{2. 1}$$

Now, we divide the displacement $\boldsymbol{u}$ into two parts, $\boldsymbol{u}_{01}$ and $\boldsymbol{u}_{02}$ which satisfy the equations

$$0=-(1-\sigma)(\rho_s-\rho)g\boldsymbol{k}+(\mathfrak{L}+\mathfrak{M})\mathrm{grad}\Theta_{01}+\mathfrak{M}\nabla^2\boldsymbol{u}_{01} \tag{2. 2}$$

$$0=(\mathfrak{L}+\mathfrak{M})\mathrm{grad}\Theta_{02}+\mathfrak{M}\nabla^2\boldsymbol{u}_{02}-\rho g \text{ grad } \varphi \tag{2. 3}$$

respectively, where

$$\Theta_{01}\equiv\mathrm{div}\, \boldsymbol{u}_{01}, \qquad \Theta_{02}\equiv\mathrm{div}\, \boldsymbol{u}_{02} \qquad (2.\ 4),\ (2.\ 5)$$

Eq. (2.2) expresses that $\boldsymbol{u}_{01}$ is the displacement caused by the apparent weight of the soil skeleton in water without external load and Eq. (2.3) means that if $\boldsymbol{u}_{02}$ is uniform in the entire body of soil, grad φ, consequently $\boldsymbol{U}$, is zero, say conversely, if pore water flows, the skeleton of the soil must be strained to that extent.

When external load and/or piezometric head φ vary with time after the initial state, the strain on the skeleton and the velocity of flow begin to leave the static state. We proceed to investigate the unsteady motion.

Disolving the quantities $\boldsymbol{u}$, Θ, $\boldsymbol{U}$ and φ into the static part $\boldsymbol{u}_0=\boldsymbol{u}_{01}+\boldsymbol{u}_{02}$, $\Theta_0=\Theta_{01}+\Theta_{02}$, $\boldsymbol{U}_0$ and φ_0 and respective deviations $\boldsymbol{u}'$, Θ', $\boldsymbol{U}'$ and φ' from the static one, we have for the deviating parts

$$(1-\sigma)\rho_s\frac{\partial^2\boldsymbol{u}'}{\partial t^2}=(\mathfrak{L}+\mathfrak{M})\mathrm{grad}\Theta'+\mathfrak{M}\nabla^2\boldsymbol{u}'-(1-\sigma)\rho g \text{ grad } \varphi' + \rho\frac{\sigma g}{k}\left(\boldsymbol{U}'-\sigma\frac{\partial\boldsymbol{u}'}{\partial t}\right) \tag{2. 6}$$

$$\frac{\partial\boldsymbol{U}'}{\partial t}+\sigma g\left\{\mathrm{grad}\varphi'+\frac{1}{k}\left(\boldsymbol{U}'-\sigma\frac{\partial\boldsymbol{u}'}{\partial t}\right)\right\}=0 \tag{2. 7}$$

$$(1-\sigma)\frac{\partial\Theta'}{\partial t}+\mathrm{div}\boldsymbol{U}'=0 \tag{2. 8}$$

We can now derive the equation for only the dilatation Θ in the following manner. In the remainder, let us omit the prime of each quantity. Taking the divergences of Eqs. (2.6) and (2.7) and the time derivative of Eq. (2.8)

$$(1-\sigma)\rho_s\frac{\partial^2\Theta}{\partial t^2}=(\mathfrak{L}+2\mathfrak{M})\nabla^2\Theta-(1-\sigma)\rho g\nabla^2\varphi+\rho\frac{\sigma g}{k}\left(\mathrm{div}\boldsymbol{U}-\sigma\frac{\partial\Theta}{\partial t}\right) \tag{2. 9}$$

$$\frac{\partial}{\partial t}\mathrm{div}\boldsymbol{U}+\sigma g\left[\nabla^2\varphi+\frac{1}{k}\left(\mathrm{div}\boldsymbol{U}-\sigma\frac{\partial\Theta}{\partial t}\right)\right]=0 \tag{2.10}$$

$$(1-\sigma)\frac{\partial^2\Theta}{\partial t^2}+\frac{\partial}{\partial t}\mathrm{div}\boldsymbol{U}=0 \tag{2.11}$$

Combining Eq. (2.10) with Eqs. (2.8) and (2.11),

$$\nabla^2\varphi=\frac{1-\sigma}{\sigma g}\frac{\partial^2\Theta}{\partial t^2}+\frac{1}{k}\frac{\partial\Theta}{\partial t} \tag{2.12}$$

Substituting from Eqs. (2.8) and (2.12) into Eq. (2.9),

$$(1-\sigma)\left\{\rho_s+\frac{1-\sigma}{\sigma}\rho\right\}\frac{\partial^2\Theta}{\partial t^2}+\frac{\rho g}{k}\frac{\partial\Theta}{\partial t}=(\mathfrak{L}+2\mathfrak{M})\nabla^2\Theta \tag{2.13}$$

This is the equation we want to derive.

As the particular case which interests us, we shall pick out the quasi-static motion. Neglecting the accelerated terms in Eqs. (2.12) and (2.13), we find

$$\frac{\partial\Theta}{\partial t}=k\nabla^2\varphi=\frac{k}{\rho g}(\mathfrak{L}+2\mathfrak{M})\nabla^2\Theta \tag{2.14}$$

Differentiating the right part of Eq. (2.14) with respect to time and inserting the left part of Eq. (2.14) into this, we make

$$\nabla^2\left[\frac{\partial\varphi}{\partial t}-\frac{k}{\rho g}(\mathfrak{L}+2\mathfrak{M})\nabla^2\varphi\right]=0 \tag{2.15}$$

This may be regarded as the basic equation of three dimensional consolidation. It is worthy of note that if $\varphi_1(t, x_i)$ is a solution of Eq. (2.15), $\varphi_1(t, x_i)+F_1(x_i)+F_2(t)$, where $\nabla^2 F_1=0$, is also the solution within the limits of quasi-static transition, in physical words, the progress of consolidation is not affected by the non-divergent flow of pore water and/or the gradual change of the water head in the entire region of the soil medium. When the non-divergent flow has already been included in the static part of the piezometric head considered previously, Eq. (2.15) is reduced to

$$\frac{\partial\varphi}{\partial t}=\frac{k}{\rho g}(\mathfrak{L}+2\mathfrak{M})\nabla^2\varphi \tag{2.16}$$

In the case of the elastic skeleton, operations $\mathfrak{L}$ and $\mathfrak{M}$ are reduced to Lame's constant λ and μ, respectively. Let us examine the special case of a column of soil supporting a load and confined in a rigid cylinder so that no lateral expansion can occur. Eq. (2.16) is then rewritten as

$$\frac{\partial\varphi}{\partial t}=\frac{k}{\rho g}(\lambda+2\mu)\frac{\partial^2\varphi}{\partial x_3{}^2}$$

By comparing this with Terzaghi's well-known equation, we can see the relation of operations $\mathfrak{L}$ and $\mathfrak{M}$ to the value a which is termed the coefficient of compressibility in soil mechanics, [5]

$$\mathfrak{L}+2\mathfrak{M}=(1+e)/a=1/(1-\sigma)a$$

Eq. (2.14) is taken as the basic equation for subsidence closely related to the flow of confined ground water[6] through the visco-elastic aquifer. We shall show it here in a simple example.

4. Example

We consider the deformation of confined aquifer caused by pumping up the ground water at a constant rate after a certain instance in a laterally infinite aquifer with a uniform depth as seen in Fig. 1.

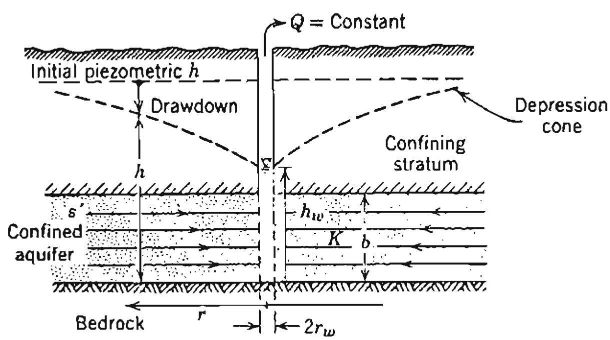

Fig. 1 Radial flow to a well completely penetrating an infinite confined aquifer with a uniform depth b and a permeability K.

Assuming that, at the initial state, the confined water had no flow and the aquifer was in equilibrium under a burden load in gravitational field, we have the equation for the deviating state from the equilibrium one

$$\frac{\partial \Theta}{\partial t}=\frac{k}{\rho g}(\mathfrak{L}+2\mathfrak{M})\nabla^2\Theta=k\nabla^2\varphi \tag{3. 1}$$

Now, supposing that the aquifer deforms only in a vertical direction and that the flow of water is uniform in the vertical cross-section and that the upper and lower boundary surfaces of the aquifer are not leaky, we can put on initial and boundary conditions,

$$t=0\ ;\ \Theta=0,\quad \varphi=0,\quad w(\equiv u_3)=0 \tag{3. 2}$$

$$t>0\ ;\ -2\pi bKr\frac{\partial\varphi}{\partial r}=Q,\ \text{at}\ r=r_w \tag{3. 3}$$

$$\Theta=0,\quad \frac{\partial\varphi}{\partial r}=0,\ \text{at}\ r=\infty \tag{3. 4}$$

$$\frac{\partial\varphi}{\partial z}=0,\ w=0,\ \text{at}\ z=0,\ \frac{\partial\varphi}{\partial z}=0\ \ \text{at}\ z=b \qquad (3.5), (3.6)$$

where, b and K are the thickness and permeability of the aquifer, respectively, r_w is the radius of the pumping well, Q is the pumping rate of water and the surface $z=0$ and b are the lower and upper boundary surfaces of the aquifer, respectively.

From the conditions (3.5) and (3.6), we can see that the quantities φ and Θ are independent of z and that the amount of subsidence is obtained by $w(z=b)=b\Theta$.

Reducing the Eq. (3.1) to ordinary differential equation by Laplace transformation, we have :

$$pV_\Theta=\frac{k}{\rho g}\left\{\mathfrak{L}(p)+2\mathfrak{M}(p)\right\}\nabla^2V_\Theta=k\nabla^2V_\varphi \tag{3. 7}$$

with

$$r=r_w\ ;\ -2\pi bKr\frac{\partial V_\varphi}{\partial r}=\frac{Q}{p} \tag{3. 8}$$

$$r=\infty\ ;\ V_\Theta=0,\quad \frac{\partial V_\varphi}{\partial r}=0 \tag{3. 9}$$

where

$$V_\Theta\equiv\int_0^\infty e^{-pt}\Theta dt,\quad V_\varphi\equiv\int_0^\infty e^{-pt}\varphi dt \qquad (3.10), (3.11)$$

Let us now consider the deformation in the Voigt model as a typical visco-elasticity. In this case, the operation is expressed by

$$\mathfrak{L}(p)+2\mathfrak{M}(p)=(\lambda+2\mu)\frac{1+cp}{1+\gamma p},\ \text{where}\ c\equiv\frac{\alpha\lambda+2\beta\mu}{\lambda+2\mu} \tag{3.12}$$

and representing by the usual notation shown in Fig. 2, the quantities λ, μ, γ and c are

$$\lambda+2\mu=\frac{E_1E_2}{E_1+E_2},\quad \gamma=\frac{\eta_2}{E_1+E_2}\quad \text{and}\quad c=\frac{\eta_2}{E_2} \tag{3.13}$$

with notice that $0<\frac{1}{c}<\frac{1}{\gamma}$. We have the solutions of Eq. (3. 7)

$$V_\varphi=\frac{\sqrt{\tau}Q}{2\pi br_w}\sqrt{\frac{1+cp}{p^3(1+\gamma p)}}\ \frac{K_0\left(\frac{\sqrt{\tau}}{\nu}rz\right)}{K_1\left(\frac{\sqrt{\tau}}{\nu}r_wz\right)} \tag{3.14}$$

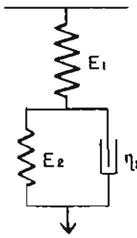

Fig. 2 Voigt's model for the rheological character of the aqufier

$$V_\theta=\frac{\sqrt{\tau}Q}{2\pi\nu b r_w}\sqrt{\frac{1+\gamma p}{p^3(1+cp)}}\;\frac{K_0\left(\frac{\sqrt{\tau}}{\nu}rz\right)}{K_1\left(\frac{\sqrt{\tau}}{\nu}r_wz\right)} \tag{3.15}$$

where K_0 and K_1 are the modified Bessel functions and

$$\nu\equiv\frac{\lambda+2\mu}{\rho g},\quad \tau\equiv\frac{\lambda+2\mu}{K\rho g}\equiv\frac{\nu}{K},$$

$$z\equiv\sqrt{\frac{p(1+\gamma p)}{1+cp}}$$

The solutions φ and Θ are determined from V_φ and V_θ by the use of the inversion theorem for the Laplace transformation

$$\varphi=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{tp}V_\varphi dp,\quad \Theta=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{tp}V_\theta dp \tag{3.16) (3.17}$$

As the functions $\sqrt{\frac{1+cp}{p^3(1+\gamma p)}}$, $\sqrt{\frac{1+\gamma p}{p^3(1+cp)}}$, $K_0\left(\frac{\sqrt{\tau}}{\nu}rz\right)$ and $K_1\left(\frac{\sqrt{\tau}}{\nu}r_wz\right)$ have the branch points at $p=0$, $-\frac{1}{c}$, $-\frac{1}{\gamma}$ and $-\infty$, the integrations in Eqs. (3.16) and (3.17) are carried out, using the contour of Fig. 3 with two cuts on the negative real axis so that the integrants are single valued functions of p within and on the contour.

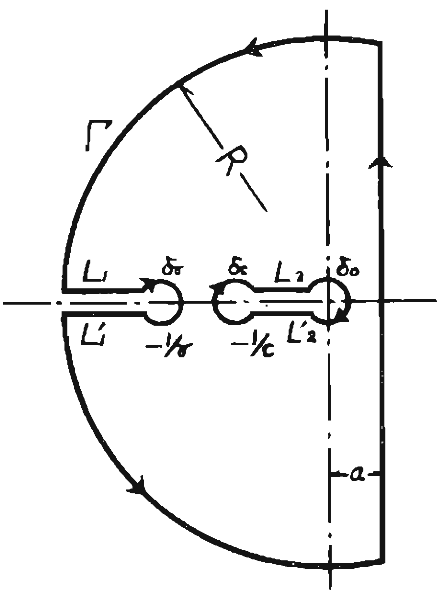

Fig. 3 The contour of inverse integration for Laplace transformations V_φ and V_θ.

In the limit as the radius of circle Γ tends to infinity and the radii of circle δ_γ and δ_c tend to zero, the respective integrals round them can be shown to vanish. On the circle δ_0, we can find the limiting value

$$\lim_{\delta_0\to 0}\frac{1}{2\pi i}\int_{\delta_0}e^{tp}V_\varphi dp$$

$$=-\frac{\tau Q}{4\pi\nu b}\left\{2C+\lim_{\delta_0\to 0}\log\left(\frac{\tau r^2}{4\nu^2}\delta_0\right)\right\} \tag{3.18}$$

as the radius δ_0 tends to zero, where $C\equiv 0.5772\cdots$ is Euler's constant.

Because the argument of p is π and $-\pi$ on the lines L_1, L_2 and L_1', L_2' respectively and then $K_0\left(\frac{\sqrt{\tau}}{\nu}rz\right)$, $K_1\left(\frac{\sqrt{\tau}}{\nu}r_wz\right)$ are expressed by the formulas,

$$K_0(\pm iz)=\mp i\frac{\pi}{2}\left[J_0(z)\mp iY_0(z)\right]$$

$$K_1(\pm iz)=-\frac{\pi}{2}\left[J_1(z)\mp iY_1(z)\right]$$

z : real, not negative,

we get

$$\frac{1}{2\pi i}\left\{\int_{L_1}dp+\int_{L'_1}dp\right\}=-\frac{\sqrt{\tau}Q}{2\pi b r_w}\;\frac{1}{\pi}\int_{1/\gamma}^{\infty}e^{-\rho t}\sqrt{\frac{1-c\rho}{\rho^3(1-\gamma\rho)}}$$

$$\times \frac{J_0(r')Y_1(r'_w)-Y_0(r')J_1(r'_w)}{J_1^2(r'_w)+Y_1^2(r'_w)}d\rho$$

and

$$\frac{1}{2\pi i}\left\{\int_{L_2}dp+\int_{L_2'}dp\right\}=-\frac{\tau Q}{2\pi b r_w}\frac{1}{\pi}\int_{\delta_0}^{1/c}e^{-\rho t}\sqrt{\frac{c\rho-1}{\rho^3(\gamma\rho-1)}}$$
$$\times \frac{J_0(r')Y_1(r'_w)-Y_0(r')J_1(r'_w)}{J_1^2(r'_w)+Y_1^2(r'_w)}d\rho$$

Finally, we get the solution for the piezometric head

$$\varphi=\frac{\tau Q}{4\pi\nu b}\left\{-2C-\lim_{\delta_0\to 0}\log\left(\frac{\tau r^3}{4\nu^2}\delta_0\right)\right\}$$
$$+\frac{\sqrt{\tau}Q}{2\pi^2 b r_w}\lim_{\delta_0\to 0}\int_{\delta_0}^{1/c}e^{-\rho t}\sqrt{\frac{1-c\rho}{\rho^3(1-\gamma\rho)}}\frac{J_0(r')Y_1(r'_w)-Y_0(r')J_1(r'_w)}{J_1^2(r'_w)+Y_1^2(r'_w)}d\rho$$
$$+\frac{\sqrt{\tau}Q}{2\pi^2 b r_w}\int_{1/\gamma}^{\infty}e^{-\rho t}\sqrt{\frac{c\rho-1}{\rho^3(\gamma\rho-1)}}\frac{J_0(r')Y_1(r'_w)-Y_0(r')J_1(r'_w)}{J_1^2(r'_w)+Y_1^2(r'_w)}d\rho \quad (3.19)$$

where $r'\equiv\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho(1-\gamma\rho)}{1-c\rho}}$, $r'_w\equiv\frac{\sqrt{\tau}}{\nu}r_w\sqrt{\frac{\rho(1-\gamma\rho)}{1-c\rho}}$

In a similar way, we can find the solution for the dilatation

$$\Theta=\frac{\tau Q}{4\pi\nu^2 b}\left\{-2C-\lim_{\delta_0\to 0}\log\left(\frac{\tau r^2}{4\nu^2}\delta_0\right)\right\}$$
$$+\frac{\sqrt{\tau}Q}{2\pi^2\nu b r_w}\lim_{\delta_0\to 0}\int_{\delta_0}^{1/c}e^{-\rho t}\sqrt{\frac{1-\gamma\rho}{\rho^3(1-c\rho)}}\frac{J_0(r')Y_1(r'_w)-Y_0(r')J_1(r'_w)}{J_1^2(r'_w)+Y_1^2(r'_w)}d\rho$$
$$+\frac{\sqrt{\tau}Q}{2\pi^2\nu b r_w}\int_{1/\gamma}^{\infty}e^{-\rho t}\sqrt{\frac{1-\gamma\rho}{\rho^3(1-c\rho)}}\frac{J_0(r')Y_1(r'_w)-Y_0(r')J_1(r'_w)}{J_1^2(r'_w)+Y_1^2(r'_w)}d\rho \quad (3.20)$$

When the radius of the pumping well is very small, we have

$$\lim_{r_w\to 0}J_1(r'_w)=0,\qquad \lim_{r_w\to 0}r_w Y_1(r'_w)=-\frac{2r_w}{\pi r_w'}=-\frac{2\nu}{\pi\sqrt{\tau}}\sqrt{\frac{1-c\rho}{\rho(1-\gamma\rho)}}$$

and then

$$\lim_{r_w\to 0}\varphi=\frac{\tau Q}{4\pi\nu b}\left[-2C-\lim_{\delta_0\to 0}\log\left(\frac{\tau r^2}{4\nu^2}\delta_0\right)-\lim_{\delta_0\to 0}\int_{\delta_0}^{1/c}\frac{e^{-\rho t}}{\rho}J_0\left(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho(1-\gamma\rho)}{1-c\rho}}\right)d\rho\right.$$
$$\left.-\int_{1/\gamma}^{\infty}\frac{e^{-\rho t}}{\rho}J_0\left(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho(1-\gamma\rho)}{1-c\rho}}\right)d\rho\right] \quad (3.21)$$

$$\lim_{r_w\to 0}\Theta=\frac{\tau Q}{4\pi\nu^2 b}\left[-2C-\lim_{\delta_0\to 0}\log\left(\frac{\tau r^2}{4\nu^2}\delta_0\right)-\lim_{\delta_0\to 0}\int_{\delta_0}^{1/c}e^{-\rho t}\frac{1-\gamma\rho}{\rho(1-c\rho)}\right.$$
$$\left.\times J_0\left(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho(1-\gamma\rho)}{1-c\rho}}\right)d\rho-\int_{1/\gamma}^{\infty}e^{-\rho t}\frac{1-\gamma\rho}{\rho(1-c\rho)}J_0\left(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho(1-\gamma\rho)}{1-c\rho}}\right)d\rho\right] \quad (3.22)$$

In the special case where $\gamma=0$, we have

$$\lim_{r_w\to 0}\varphi=\frac{\tau Q}{4\pi\nu b}\left[-2C-\lim_{\delta_0\to 0}\log\left(\frac{\tau r^2}{4\nu^2}\delta_0\right)\right.$$
$$\left.+\lim_{\delta_0\to 0}\int_{\delta_0}^{1/c}\frac{\partial}{\partial\rho}W(\rho t)J_0\left(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho}{1-c\rho}}\right)d\rho\right]$$
$$=\frac{Q}{4\pi bK}\left[-2C-\lim_{\delta_0\to 0}\log\left(\frac{\tau r^2}{4\nu^2}\delta_0\right)-\lim_{\delta_0\to 0}W(\delta_0 t)\right.$$
$$\left.+\frac{\sqrt{\tau}}{\nu}r\int_0^{1/c}W(\rho t)\frac{\partial}{\partial\rho}\sqrt{\frac{\rho}{1-c\rho}}\cdot J_1\left(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho}{1-c\rho}}\right)d\rho\right]$$
$$=\frac{Q}{4\pi bK}\left[-C-\log\left(\frac{\tau}{4\nu^2}\frac{r^2}{t}\right)+\frac{\sqrt{\tau}}{\nu}r\int_0^{\infty}W\left(\frac{t\xi^2}{c\xi^2+1}\right)J_1\left(\frac{\sqrt{\tau}}{\nu}r\xi\right)d\xi\right] \quad (3.23)$$

where $W(x)\equiv\int_x^\infty \frac{e^{-\eta}}{\eta}d\eta$ is the "well function" and $\xi\equiv\sqrt{\frac{\rho}{1-c\rho}}$,

and also

$$\lim_{r_w\to 0}\Theta=\frac{\tau Q}{4\pi\nu^2 b}\Big[-2C-\lim_{\delta_0\to 0}\log\Big(\frac{\tau r^2}{4\nu^2}\delta_0\Big)-\lim_{\delta_0\to 0}\int_{\delta_0}^{1/c}\frac{e^{-\rho t}}{\rho}J_0\Big(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho}{1-c\rho}}\Big)d\rho$$
$$-c\int_0^{1/c}\frac{e^{-\rho t}}{1-c\rho}J_0\Big(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{\rho}{1-c\rho}}\Big)d\rho\Big]$$
$$=\frac{\rho g Q}{4\pi b(\lambda+2\mu)K}\Big[-C-\log\Big(\frac{\tau r^2}{4\nu^2 t}\Big)+\frac{\sqrt{\tau}}{\nu}r\int_0^\infty W\Big(\frac{t\xi^2}{c\xi^2+1}\Big)J_1\Big(\frac{\sqrt{\tau}}{\nu}r\xi\Big)d\xi$$
$$-e^{-t/c}\int_0^1\frac{e^{-\frac{t}{c}\eta}}{\eta}J_0\Big(\frac{\sqrt{\tau}}{\nu}r\sqrt{\frac{1-\eta}{c\eta}}\Big)d\eta\Big] \qquad ;\ (1-c\rho\equiv\eta)$$
$$=\frac{\rho g Q}{4\pi b(\lambda+2\mu)K}\Big[-C-\log\Big(\frac{\tau r^2}{4\nu^2 t}\Big)+e^{-t/c}W\Big(-\frac{t}{c}\Big)$$
$$+\frac{\sqrt{\tau}}{\nu}r\int_0^\infty W\Big(\frac{t\xi^2}{c\xi^2+1}\Big)J_1\Big(\frac{\sqrt{\tau}}{\nu}r\xi\Big)d\xi$$
$$-\frac{\sqrt{\tau}}{\nu}re^{-t/c}\int_0^\infty W\Big\{-\frac{t}{c(c\xi^2+1)}\Big\}J_1\Big(\frac{\sqrt{\tau}}{\nu}r\xi\Big)d\xi\Big] \tag{3.24}$$

Using the dimensionless time t^* and distance r^*

$$t^*\equiv\frac{t}{c},\qquad r^*\equiv\sqrt{\frac{\tau}{c}}\,\frac{r}{\nu}, \tag{3.25}$$

the amount of subsidence ζ is obtained by

$$\zeta=b\Theta$$
$$=\frac{\rho g Q}{4\pi(\lambda+2\mu)K}\Big[-C-\log\Big(\frac{r^{*2}}{4t^*}\Big)+e^{-t^*}W(-t^*)$$
$$+\int_0^\infty W\Big(\frac{t^*y^2}{y^2+1}\Big)r^*J_1(r^*y)dy$$
$$-e^{-t^*}\int_0^\infty W\Big(\frac{-t^*}{y^2+1}\Big)r^*J_1(r^*y)dy\Big]$$
$$=\frac{\rho g Q}{4\pi(\lambda+2\mu)K}\Big[-C-\log\Big(\frac{r^{*2}}{4t^*}\Big)+e^{-t^*}W(-t^*)$$
$$-\int_0^\infty\Big\{C+\log\Big(\frac{t^*y^2}{y^2+1}\Big)+\sum_{n=1}^\infty\frac{(-1)^n}{n\cdot n!}\Big(\frac{t^*y^2}{y^2+1}\Big)^n\Big\}r^*J_1(r^*y)dy$$
$$+e^{-t^*}\int_0^\infty\Big\{C+\log\Big(\frac{t^*}{y^2+1}\Big)+\sum_{n=1}^\infty\frac{1}{n\cdot n!}\Big(\frac{t^*}{y^2+1}\Big)^n\Big\}r^*J_1(r^*y)dy\Big]$$
$$=\frac{\rho g Q}{4\pi(\lambda+2\mu)K}\Big[-C-\log\Big(\frac{r^{*2}}{4t^*}\Big)+e^{-t^*}W(-t^*)-(C+\log t^*)$$
$$-2r^*\int_0^\infty\log y\cdot J_1(r^*y)dy+\int_0^\infty\frac{2y}{y^2+1}J_0(r^*y)dy$$
$$-\sum_{n=1}^\infty\frac{(-1)^n}{n\cdot n!}t^{*n}\int_0^\infty\frac{d}{dy}\Big\{1-\frac{1}{y^2+1}\Big\}^n J_0(r^*y)dy$$
$$+e^{-t^*}(C+\log t^*)-e^{-t^*}\int_0^\infty\frac{2y}{y^2+1}J_0(r^*y)dy$$
$$+e^{-t^*}\sum_{n=1}^\infty\frac{1}{n\cdot n!}t^{*n}\Big\{1+\int_0^\infty\frac{d}{dy}\Big(\frac{1}{y^2+1}\Big)^n J_0(r^*y)dy\Big\}\Big]$$
$$=\frac{\rho g Q}{4\pi(\lambda+2\mu)K}\Big[-2C-\log\frac{r^{*2}}{4}+e^{-t^*}W(-t^*)-2\Big(-C-\log\frac{r^*}{2}\Big)$$

$$+2(1-e^{-t^*})K_0(r^*)+e^{-t^*}\Big(C+\log t^*+\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\,t^{*n}\Big)$$

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,t^{*n}\Big\{\sum_{m=0}^{n-1}\frac{(-1)^m}{m!(m+1)!(n-m-1)!}\Big(\frac{r^*}{2}\Big)^{m+1}K_{m+1}(r^*)\Big\}$$

$$-2e^{-t^*}\sum_{n=1}^{\infty}\frac{1}{(n!)^2}\,t^{*n}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\Big]$$

$$=\frac{\rho gQ}{2\pi(\lambda+2\mu)K}\Big[(1-e^{-t^*})K_0(r^*)+\sum_{n=1}^{\infty}\frac{1}{(n-1)!\,n!}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\sum_{m=0}^{\infty}\frac{(-1)^m t^{n+m}}{m!(n+m)}$$

$$-e^{-t^*}\sum_{n=1}^{\infty}\frac{1}{(n!)^2}\,t^{*n}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\Big]$$

$$=\frac{\rho gQ}{2\pi(\lambda+2\mu)K}\Big[(1-e^{-t^*})K_0(r^*)+\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\Big\{1-e^{-t^*}\sum_{m=0}^{n}\frac{t^m}{m!}\Big\}\Big]$$

Summarizing the result of the calculation, we have the solutions, φ, Θ and ζ in a special case where $\mathfrak{L}+2\mathfrak{M}=(\lambda+2\mu)\,(1+c\frac{\partial}{\partial t})$

$$\lim_{r_w\to 0}\varphi=\frac{Q}{2\pi bK}\Big[K_0(r^*)+\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\Big\{1-e^{-t^*}\sum_{m=0}^{n-1}\frac{t^{*m}}{m!}\Big\}\Big] \tag{3.26}$$

$$\lim_{r_w\to 0}\Theta=\frac{\rho gQ}{2\pi(\lambda+2\mu)bK}\Big[\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\Big\{1-e^{-t^*}\sum_{m=0}^{n}\frac{t^{*m}}{m!}\Big\}\Big] \tag{3.27}$$

$$\lim_{r_w\to 0}\zeta=-\frac{\rho gQ}{2\pi(\lambda+2\mu)K}\Big[\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{r^*}{2}\Big)^nK_n(r^*)\Big\{1-e^{-t^*}\sum_{m=0}^{n}\frac{t^{*m}}{m!}\Big\}\Big] \tag{3.28}$$

where
$$t^*\equiv\frac{t}{c},\quad r^*\equiv\sqrt{\frac{\tau}{c}}\,\frac{r}{\nu}\equiv\sqrt{\frac{\rho g r^2}{cK(\lambda+2\mu)}} \tag{3.29}$$

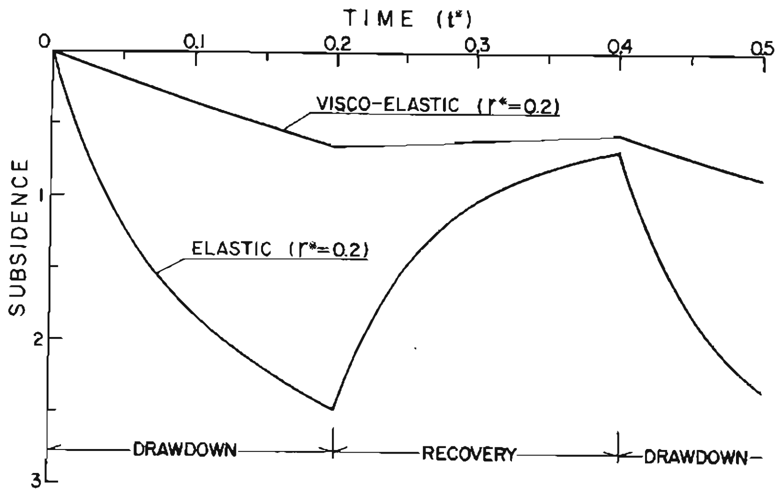

Fig. 4 Comparison between the deformations of elastic and visco-elastic confined aquifers in a case where $r^*=0.2$ and $\frac{\rho gQ}{4\pi(\lambda+2\mu)K}=1$ for pumping up the water intermittently.

Comparison between the elastic and visco-elastic deformations is shown in Fig. 4 with the exact solution $\zeta_{ela}=\frac{\rho gQ}{4\pi(\lambda+2\mu)K}\cdot W\left(\frac{r^{*2}}{4t^{*}}\right)$ in the elastic subsidence and the approximation of the third power of t^{*m} in visco-elastic one, Eq (3.28) in a case where $r^{*}=0.2$ and $\frac{\rho gQ}{4\pi(\lambda+2\mu)K}=1$ for pumping up the water intermittently.

Acknowledgement

The author wishes to express his sincere appreciation to professors S. Okuda and H. Kunish of Kyoto University for their cordial encouragement.

His thanks are also due to Mr. K. Kitaoka, member of the staff of Seiken-Reiki Co., Ltd. and Miss A. Miyagami for their whole-hearted help.

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